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A Dynamic Model for
Modern Military Conflict

by

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In order to account for the importance of accurate and timely information in modern warfare, four state variables are defined corresponding to information and forces of two opposing sides. The dynamics of the interactions between the variables is modeled by non-linear evolution equations.

The arbitrary non-linear attrition functions are approximated by polynomials of second degree. Of the 32 possible coefficients, 20 are identifiable with C3I, counter-C^3, intelligence, and firepower of the opposite sides. The remainder are set to zero and the resulting system of dynamical equations are examined for stationary points and for stability. It is determined that several stationary points are possible and a method is presented to determine one of them as the solution of a system of linear equations.

The model is examined for stability near this equilibrium point. It is shown to be environmentally unstable, and to have a bifurcation. That is, when unstable, either side may win depending on whom gains an initial advantage. When stable at unity equilibrium (equal forces in the field) a force multiplication ratio is defined by the ratio of the force replenishment rates. This ratio is easily calculated from the C^3I, counter-C^3, intelligence, and firepower parameters of the model.

Examples of stable and unstable force and information evolutions are presented.

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Paul H. Moose

I. Introduction

In modern warfare, an operational commander is intimately concerned with the quality, timeliness and completeness of his "picture" of the tactical situation. To a very large extent, his fortunes and those of his assigned forces depend on his having available, when and where he needs it, accurate data about the status, location and activities of both his own and the enemies' forces. Similar requirements extend well down into subordinate echelons of his command, including individual unit commanders and even "smart" weapons.

Today, the methods by which information is acquired are remarkably diverse. Sophisticated radar and electronic intercept equipments and a variety of imaging and acoustic sensors on fixed, mobile, airborne and satellite platforms send reports to the command center. Inputs from direct observation of the commander's own personnel along with reports from special intelligence are communicated, by a variety of means, to the command center too. All this creates a massive and continuing informational input to the commander and his staff. The staff, assisted by modern automatic data processing equipments, is regularly creating and updating their assessment of the situation in order to give the best operational picture they can to their commander. The commander will, to a very great degree, make rational and reasonably predictable decisions for the future activities of his forces based on the world view he has developed from this sequence of images.

There are several important obsevations to make about this "image of reality" that the commander works with. First, the images he has are never absolutely correct, that is, they contain errors. Nor are they perfectly sharp, that is, there are always many questions that are unanswered, or elements of contradiction or ambiguity. Secondly, a given image grows more and more "fuzzy" the further into the future one attempts to extrapolate it. This is because most elements of the picture are dynamic, that is, they change (location, behavior, etc.) with time. Some attributes of the elements may be partially constrained. For example, a ship cannot move faster than about 30 knots. Nevertheless, after sufficient time elapses most features of the picture will have total freedom to take on any of their possible values or conditions.

This "fuzzyness in the crystal ball" axiom has a corollary. If the commander loses, or turns off, his sensors or sources of information, his "current image" will grow fuzzier and fuzzier with time until it is eventually completely blurred. Put another way, a commander only maintains his uncertainty about what is going on below its worst possible level by virtue of the continual application of systemic resources to guarantee an inflow of new information. Thus, sensor devices and information sources provide constraints on uncertainty. They do this by continually importing information to offset uncertainty's inevitable growth.

But merely gathering new information from one's own assets to combat the growth of uncertainty may not tell the whole story. For example, Rona [2] has described the concept of "information war" as a dominant factor in the conduct of modern warfare. In an information war, one

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(2) Rona, T. P., "Weapon Systems and Information War", Boeing Aerospace Co., Seattle, WA, July 1976.

actively attempts to deny the enemy knowledge of his force positions, numbers, intentions, etc. This is done by a variety of means. Included, for example, are cover and deception tactics, distribution of radar chaff, decoys, false messages, etc. One also works to keep his own communications intact and secure, but to intercept, exploit and/or jam those of the enemy. One may also try to physically disable enemy C^3 facilities and channels. In all of this, the purpose is to try to reduce one's own uncertainty by assuring a steady, reliable inflow of relevant information, a term we have already described above. But moreoever, to disrupt the opposition's flow of information and ultimately blur or distort his image of the operational situation. This will cause, we maintain, poor decisions on the enemy's part thereby reducing his force effectiveness.

So far we have not mentioned directly the role of the forces themselves. Although it is undeniable that accurate, timely information and reliable rapid communications are essential ingredients for success in battle, they must be coupled to effective fighting units in order to have any real utility. Just as good management may be essential to a successful business, management alone cannot realize any true results without suitable raw materials and an appropriate workforce.

In military combat this translates into trained men, adequate transport and effective weapons. All other factors equal, we would expect the side with the largest force to prevail, if not in each individual battle, at least in the overall campaign or war. Note the emphasis on all other factors being equal. In general, we expect asymmetries, perhaps major ones, in the area of command, and control, communications, intelligence, deception, electronic warfare and even in tactical doctrine and perhaps strategic objectives.

At least since the time of Lanchester [3] , military planners, historians and analysts have been interested in analytical models of combat. With the advent of high speed digital computers, a number of quite detailed combat simulations and wargames have evolved. But it has been difficult to account for the effects of C^3I on the outcomes of conflicts in spite of a realization of its critical nature. For example, Mr. Andrew Marshall, DoD's Director of Net Assessment, said in 1977, that, "theater models have been assessed to have virtually no utility, particularly since they lack the ability to treat the major asymmetries that exist on both sides in tactical doctrine and the structures of command and control. In addition, the failure of present theater models to account for certain factors (surprise, deception, leadership, etc.) that historically have permitted a force that is inferior in number and equipment to defeat a superior one does not inspire confidence in the use of such models" [4] . In 1980, D.P. Gaver [5] demonstrated the dependence of statistics of force attrition on the information available to the forces and/or their weapons, (accuracy of missile targeting data, for

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(3) Taylor, J.C., "A Tutorial on Lanchester-Type Models of Warfare (U)", Proceedings 35th Military Operations Research Symposium, July 1975.

(4) Theater Level Gaming and Analysis Workshop for Force Planning, Lawrence T. Low, Vol II, 1981, SRI International, Menlo Park, CA, Contract N00014-77-C-0129.

(5) Gaver, D.P., "Models that Reflect the Value of Information in a Command and Control Context", Naval Postgraduate School, Monterey, CA, NPS-55-80-027, Oct 1980.

example). But we also note that the fortunes of the forces will, in turn, have an effect on the ability of the C^3I system to function effectively, not only through the fraction of the forces that remain well coordinated and informed, but also through the portion of total systemic assets that can be devoted to C^3I and counter-C^3 tasks.

What appears as an inevitable consequence of modern military evolution toward greater reliance on smart weaponry, extensive intelligence and surveillance gathering sensors and high speed, high volume communications nets is an analytical requirement to simultaneously model the time varying behavior of the status of both the information capabilities and the force levels of each side. These four quantities are all coupled. Together they define the status of a conflict at any point in time.

The development and presentation of such a model, along with an illustration of certain important patterns of behavior that this four-species system can exhibit, is the objective of the remainder of our present report.

II. Aggregate Conflict Modeling

From the Introduction, it is apparent that we wish to consider the interaction of information (lack of uncertainty) and forces (men and materials) on each of two sides (we shall refer to them as side X and side Y) engaged in conflict. That is, their mutual objective will be the attrition of each other's assets and the defense of their own. Schematically, the interactions between the four system variables and the independent system inputs are illustrated in Figure 2.1.

The matter of units in which the variables are measured needs explanation. As far as the forces are concerned, we break no new ground. According to Morse and Kimball [6]; "Each side has at any moment, a certain number of trained men, of ships, (sic) planes, tanks, etc., which can be thrown into battle in a fairly short time, as fast as transport can get them to the scene of action. The total strength of the force is determined by the effectiveness of each component part. At any stage of war, we can say that a ship is as valuable as so many armies, that a submarine is as valuable as so many squadrons of planes, etc. To this crude approximation, each unit can be measured in forms of some arbitrary unit - so many equivalent army divisions for instance." Morse and Kimball go on to recognize that this is an oversimplification due to qualitative and situational differences between air, sea, land, armor, etc., but that this type of simplification is in the nature of constructing macroscopic models of this sort (in their case, Lanchester Models). We are forced to adopt their same point of view, as indeed are all macroscopic modelers of complex phenomena, with regard to the forces.

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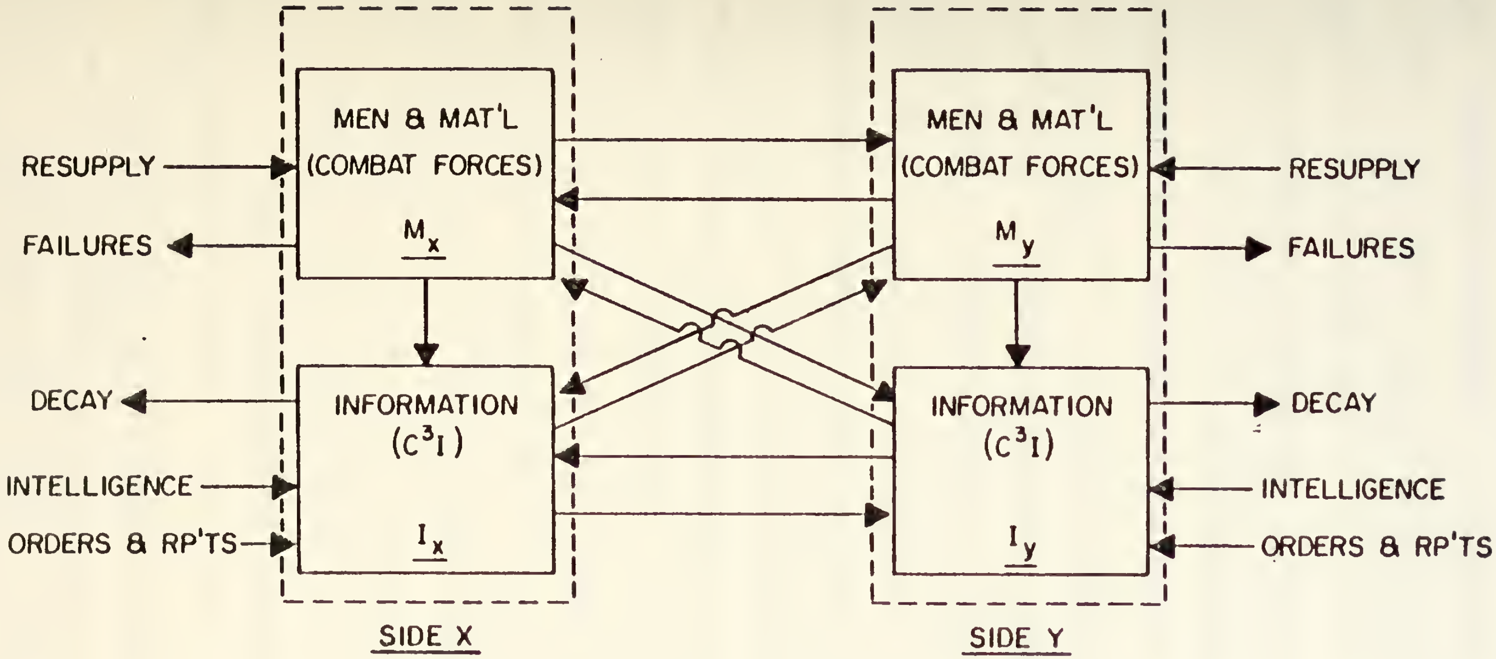
(6) Morse, P. and Kimball, Methods of Operations Research, the MIT Press, Cambridge, Mass, 1951.



A FOUR SPECIES
MODEL OF
MODERN MILITARY CONFLICT

Figure 2.1

No less so is our need for aggregation with regard to modeling the total state of information at any moment in time as it exists in the "images of reality" perceived by all the commanders that are taking decisions for allocation of their forces. The value of "knowing" an enemy's strength and location must be placed on a common scale with "knowing (correctly)" his intentions as well as being deceived, e.g. "knowing incorrectly" his intentions. The matter of measuring all of the elements of information about a conflict on a common scale, or in common units, if you will, strikes at one of the central problems in C^3I modeling and evaluation, namely, determining the relative utility of various types of information. Although we recognize the extreme simplifying nature of our approach, we must steadfastly maintain that it is possible to conceive of an overall state of (accurate) knowledge that each side has about the war at each given instant of time and that the commander's decisions, and thusly the forces' performance of their assigned missions, will on the average improve as this overall knowledge is more complete, but will lose in effectiveness as it is less complete.

Since we clearly conceive of information as a lack of uncertainty about the environment we may as well elect to measure it, on an macroscopic level, in bits. It remains to determine the relative number of bits contributed by various elements of data but this in principle seems no more an improbable feat than determining the relative worth of a submarine on the same scale with a squadron of helicopters.

Let us look again now at Figure 2.1. The four aggregate or macroscopic variables that determine the state of the conflict at each instant of time are M_x, I_x, M_y and I_y the forces and information, measured on the aforementioned scales of units, of sides X and Y respectively. Each of these is not to be just a constant number, but changes during the course of the conflict, that is, each is a function of time $M_x(t)$, $I_x(t)$, $M_y(t)$ and $I_y(t)$. The units of the time scale need not be chosen to be seconds; it may be more appropriate to measure time in minutes, hours, days or even weeks. Since time is to be a continuous independent variable, its units are really immaterial, however, we shall elect for the balance of this paper to think of it in hours.

It is not really the absolute values of the state variables that interest us in modeling as much as their rates of change. This tells us whether side X or side Y is gaining or losing in knowledge and forces at any given moment. It is clear from our diagram in Figure 2.1 that the effect of side X being continually resupplied is to cause M_x to increase (have a positive first derivative). However, we also expect some failures. Equipment will break down, some men will grow ill or be incapacitated in training, etc., causing M_x to decrease (have a negative first derivative). Side X may also elect to divert some men and supplies to strictly intelligence gathering work or men and supplies to C^3 support functions, similarly depleting M_x. These gains and loses are internal to side X and independent of the actions taken by side Y against X.

However, we also expect side X to lose men and materials due to the war efforts of side Y. This is combat related attritions and we presume it will depend both on the size of side Y's force, M_y and on his state of knowledge about the conflict which we are modeling with the macroscopic variable I_y. This is indicated by the arrows from M_y and I_y to M_x in Figure 2.1.

In addition, we want to account for side Y's counter-C^3 efforts against X. These are of two types. Physical destruction of X's C^3I facilities and assets is one. These activities are undertaken by either specifically designated or regular components of $Y^{'s}$ forces. Whichever, we presume that the amount of effort Y can devote to these physical counter-C^3 activities will be a fraction (perhaps small), of his total force. The second type of counter-C^3 depends directly on $Y^{'s}$ information about $X^{'s}$ intelligence and C^3 system. This type of counter-C^3 includes deceptions, misinformation, jamming of communications and radars, decoys, etc. These two types of counter-C^3 are indicated in Figure 2.1 by the arrows from M_y and I_y to I_x. That is, we expect counter-C^3 activities will tend to reduce $X^{'s}$ knowledge about and ability to control the overall environment.

However, totally independent of $Y^{'s}$ efforts, $X^{'s}$ information about the environment grows old, and since the environment is dynamic (things are always changing), $X^{'s}$ uncertainty tends to grow with time (information decays) as we described previously in the Introduction. These natural losses, as well as counter-C^3 information losses are offset by "informational replenishment". This is the purpose of intelligence, surveillance, battlefield reports, etc., as well as orders from higher command levels that re-focus the commander's mission and responsibility. These "natural losses" and external "informational replenishments" are indicated in the diagram of Figure 2.1 as external inputs to and outputs from I_x.

We have described the phenomena that are to be accounted for in the model from $X^{'s}$ viewpoint. If we desire to model two modernly equipped and organized opponents, then all the same may be said for side Y and $X^{'s}$ efforts against Y. This "bilateral" structure is indicated in Figure 2.1.

To summarize this section:

A. System State Variables

1.) I_x : Side X's Information Level

2.) M_x : Side X's Force Level

3.) I_y : Side Y's Information Level

4.) M_y : Side Y's Force Level

B. Independent Variable

1.) Time : hours (arbitrary, can be seconds, minutes, days, weeks, etc.)

C. Replenishment

1.) Intelligence reports, battlefield reports, surveillance reports, orders, etc.

2.) Resupply of men, weapons, transport, etc.

D. Natural Losses

1.) Increased uncertainty as information becomes dated.

2.) Equipment breakdown, troop illness and training casualties, etc.

E. Combat Losses of Information

1.) Losses due to physical attacks on C^3I assets.

2.) Losses due to information war: jamming, deception, etc.

F. Combat Losses of Forces

1.) Attrition due to firepower of opposition. Effectiveness dependent on opposition's C^3I as well as his force level.

III. Evolution Equations

Evolution problems are common to all the sciences; life, physical and social/economic. Evolution theory in modern science is a mathematical body of knowledge that can be used to model complicated real life problems. Non-linear mechanics, multiple compound chemical reactions, urban development, population biology, ecosystems, economic growth, industrial development and growth of pollutants, cell biology and genetic evolution are some of the many important complex system problems to which evolution theory has been applied.

The mathematical basis is to characterize the particular problem with a system of n, non-linear first order differential equations. The most general form of such a structure is,

$$\dot{\underset{\sim}{S}}(t) = \underset{\sim}{F}\,(S(t),\ \underset{\sim}{\mu}) + \tilde{\underset{\sim}{Q}}(t) \qquad (3\text{-}1)$$

where $\underset{\sim}{S}(t)$ is an nx1 column vector of the system state variables, $\dot{\underset{\sim}{S}}(t)$ are the first derivatives of the state variables and $\underset{\sim}{F}(\underset{\sim}{S}(t),\ \underset{\sim}{\mu})$ is an nx1 column vector of arbitrary growth (or attrition) functions. Each element of $\underset{\sim}{F}$ depends in some known (or modeled) way on all the state variables and on a set of parameters $\underset{\sim}{\mu}$. The dependence on the other state variables may be, and usually is, non-linear. The nx1 column vector $\tilde{\underset{\sim}{Q}}(t)$ is the "drive" or "input" to the system, and is assumed independent of the state variables $\underset{\sim}{S}$. If the parameters $\underset{\sim}{\mu}$ are not functions of time, the system is called "autonomous". If they are functions of time, it is called "non-autonomous". [7]

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(7) Iooss, G. and Joseph, D.D., Elementary Stability and Bifurcation Theory, Springer-Verlag, 1980.

In our model for modern military conflicts, $\underset{\sim}{S}$ is the 4x1 column vector $\underset{\sim}{S} = [I_x\ M_x\ I_y\ M_y]^T$ of system state variables. $\underset{\sim}{\tilde{Q}}$ is the 4x1 column vector of replenishment rates, $\underset{\sim}{\tilde{Q}} = [\tilde{Q}_x\ \tilde{R}_x\ \tilde{Q}_y\ \tilde{R}_y]^T$. F_1, F_2, F_3, F_4 are the attrition functions for side $X^{'s}$ information, side $X^{'s}$ forces, side $Y^{'s}$ information and side $Y^{'s}$ forces respectively. These attrition functions are to account for natural losses and for losses (or growth) induced by the other three system variables.

We shall presume that our system can be modeled by attrition functions that are at most quadratic. That is, each one shall be of the form

$$F_i\ (\underset{\sim}{S}\ ,\ \underset{\sim}{\mu}) = -S_i \sum_{j=1}^{n} \alpha_{ij}\ S_j - \sum_{j=1}^{n} \tilde{a}_{ij}\ S_j \tag{3-2}$$

to include product terms involving the i^{th} system variable with all the n system variables plus linear terms in all the n system variables.

Loss functions of the form of (3-2), not including the linear terms, are known as Lotka-Volterra loss functions or sometimes just Volterra functions and are commonly encountered in ecosystem models [8]. The parameter set $\underset{\sim}{\mu}$ consists of the set of $2n^2$ coefficients $\{\alpha_{ij}\}$ and $\{\tilde{a}_{ij}\}$ for $i=1,2,..n$ and $j=1,2,..n$. Thus our four-species model is characterized by at most 32 parameters.

---

(8) May, R.M., Stability and Complexity in Model Ecosystems, Princeton Univ Press, 1974, 2nd Edition.

IV. A Specific Model Proposed

We wish now to construct a model within the generalized form of the Lotka-Volterra equations, that can account for the important interactions of modern military conflicts described in sections I and II. The specific system of equations we shall investigate are the following:

$$\left.\begin{aligned}
\dot{I}_x &= I_x[\alpha_{xy}I_y + \gamma_{xy}M_y] - [\tilde{\alpha}_x I_x - c_{xx}M_x] + \tilde{Q}_x \\
\dot{M}_x &= - M_x[\alpha_{xy}I_y + \beta_{xy}M_y] - [\tilde{b}_x M_x + \tilde{d}_{xx}I_x + \tilde{d}_{xy}I_y + \tilde{b}_{xy}\, M_y] + \tilde{R}_x \\
\dot{I}_y &= - I_y[\alpha_{xy}I_x + \gamma_{yx}M_x] - [\tilde{a}_y I_y - \tilde{c}_{yy}M_y] + \tilde{Q}_y \\
\dot{M}_y &= - M_y[\delta_{yx}I_x + \beta_{yx}M_x] - [\tilde{b}_y M_y + \tilde{d}_{yy}I_y + \tilde{d}_{yx}I_x + \tilde{b}_{yx}\, M_x] + \tilde{R}_y
\end{aligned}\right\} \quad (4\text{-}1)$$

These equations are seen to be of the generalized Lotka-Volterra form; however, only 20 of the 32 parameters that might possibly be included have been retained. The particular interactions that we have modeled are designated as "Conflict Model V".

A word on the notation. Greek letters designate coefficients of quadratic terms in the model; english letters designate coefficients of linear terms. Subscripts x and y are used to identify the interaction variables, the first indicating the side suffering the attrition and the second indicating the side that is the source of the attrition. For example, β_{xy} indicates the area fire (quadratic) attrition coefficient of side $Y^{'s}$ forces against side $X^{'s}$ forces. A complete list of all 20 parameters plus the four source terms is presented in Table 4.1. The model has been constructed with algebraic signs so that all the parameters are presumed to be non-negative.

Given an initial set of force values and a set of parameters, the solution for this system of equation will define the time history of knowledge and forces as the conflict between the two sides evolves.

Since there is nothing about the nature of the equations to prevent negative values for the state variables, we must stop our solution when one or more of the variables reaches zero. We can either declare the conflict over, or continue the evolution with a new system.

Determining actual values for the parameters listed in Table 4.1 may be a difficult task for actual or potential conflicts. More will be

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Source Terms

$\tilde{R}_x$ = X's information input rate

$\tilde{R}_x$ - X's men, weapons and material resupply rate

$\tilde{Q}_y$ - Y's informational input rate

$\tilde{R}_y$ - Y's men, weapons and material resupply rate

Interaction Coefficients for Information Evolution

$\alpha_{xy}(\alpha_{yx})$ - Deception, jamming, decoy, etc; counter-C^3 effectiveness of Y's (X's) information assets against X's(Y's) information system.

$\gamma_{xy}(\gamma_{yx})$ - Sabotage, command post attacks, communications attacks, etc; counter-C^3 effectiveness of Y's(X's) force assets against X's(Y's) informational assets.

$\tilde{a}_x(\tilde{a}_y)$ - X's(Y's) natural rate of information loss per bit

$\tilde{c}_{xx}(\tilde{c}_{yy})$ - X's(Y's) rate of information gain per unit of forces devoted to information producing activities.

Interaction Coefficients for Force Evolution

$\delta_{xy}(\delta_{yx})$ - Increase in loss rate of X's(Y's) force due to Y's(X's) knowledge; quadratic C^3 effectiveness coefficient.

$\beta_{xy}(\beta_{yx})$ - X's(Y's) loss rate of forces due to Y's(X's) force size (Lanchester "area" fire coefficient).

$\tilde{b}_x$ $(\tilde{b}_y)$ - X's(Y's) normal loss rate of men and equipment due to illness, accident, equipment failures, etc.

$\tilde{d}_{xx}(\tilde{d}_{yy})$ - X's(Y's) loss rate of men and materials due to diversion to intelligence and C^3 producing activities for own side (see $\tilde{c}_{xx}(\tilde{c}_{yy})$).

$\tilde{d}_{xy}(\tilde{d}_{yx})$ - Increase in loss rate of X's(Y's) force due to Y's(X's) knowledge; linear C^3 effectiveness coefficient.

$\tilde{b}_{xy}(\tilde{b}_{yx})$ - X's(Y's) loss rate of forces due to Y's(X's) force size (Lanchester "aimed" fire coefficient).

Table 4.1

Definition of Model V Parameters.

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said about this problem in the final section. But the real value of a mathematical model lies in its ability to analytically predict the types of behavior that can be exhibited, and in particular to determine if this behavior is particularly sensitive to slight variations in one or more of the parameters.

It is now well known [May, op cit], the more species interacting in a system of evolution equations, the more prone the system is to unstable behavior. This relationship between complexity and stability is an important one for us. In an earlier paper [9] modeling the dynamics of information war with a two-species model (equivalent to I_x and I_y of the current model), it was shown that the system was ultra-stable (also called environmentally

(9) Moose, Paul H., "A Dynamic Model for C^3 Information Incorporating the effects of Counter C^3", NPS Report 62-81025PR, Dec 1980.

stable). That is, the system always returned to its equilibrium or steady state conditions, when perturbed from that conditions, regardless of the values of the interaction coefficients. We shall see that such is not the case for the four-species (I_x, M_x, I_y, M_y) model we propose here. Furthermore, the information war two-species model had but two equilibrium points in its two-dimensional state space, and we were able to show that only one of the two could lie inside the physically accessible region of the state-space. In a four-species model of the Lotka-Volterra type, there is a potential for 16 equilibrium points in the four-dimensional state space. (Because not all interactions are retained, Conflict V has 12 equilibrium points). The location of these equilibrium points, their sensitivity to parameter variations, and the dynamic behavior of the system in the neighborhood of the equilibrium points is the subject of the analysis in the next section.

V. Equilibria and Stability

An important mathematical feature of the evolution equations are their "stationary" or equilibrium" points. These points are sets of values for the state variables for which the net rates of change are simultaneously zero. If the system ever reaches one of these points it will no longer evolve without some external intervention.

If the system were linear, there would only be one equilibrium point. In the theory of linear systems, the values of the state variables at equilibrium are associated with the "steady state" response of the system to the constant forcing functions, $\tilde{\underset{\sim}{Q}}$. However, since our model is non-linear, it may have multiple equilibria. An n^{th} order Lotka-Volterra model, which is quadratic, has 2^n equilibrium vectors if all interaction terms are present. Thus, a four-species model has at most 16 distinct equilibrium vectors. In Model V, there are at most 12 because only 20 of the 32 possible interaction terms seemed to have physical interpretation for coupling the informational and force variables.

Determining the equilibrium points is in general a very difficult problem. For a given set of coefficients, some of the equilibrium vectors might have negative and or imaginary elements. These points would be inaccessible to a real system. Therefore, the only equilibria of interest to us are those that have all real, positive elements.

There is a technique by which we may determine one equilibrium as the solution of a system of linear equations. The method is derived for the general n^{th} order Lotka-Volterra model in Appendix A. We shall derive it here explicitly for the four-species system, Model V that has been proposed as a model of military conflict in the previous section.

We begin by assuming there is an equilibrium point which we designate $\{I_{xe}, M_{xe}, I_{ye}, M_{ye}\}$. If these values are substituted into Eqn (4-1), the right sides are zero and hence all the rates of change are simultaneously zero and the system is at a stationary or equilibrium point. In the same equations, let us redefine the resupply rates and linear coefficients as follows:

$$\left.\begin{array}{llllllllll}
\tilde{Q}_x = Q_x I_{ye} & & , & \tilde{a}_x = a_x I_{xe} & & & , & \tilde{c}_{xx} = c_{xx} I_{xe} \\
\tilde{R}_x = R_x M_{xe} & , \ \tilde{b}_x = b_x M_{xe} & , & \tilde{d}_{xx} = d_{xx} M_{xe} & , & \tilde{d}_{xy} = d_{xy} M_{xe} & , & \tilde{b}_{yx} = b_{xy} M_{xe} \\
\tilde{Q}_y = Q_y I_{ye} & & , & \tilde{a}_y = a_y I & & & , & \tilde{c}_{yy} = c_{yy} I_{ye} \\
\tilde{R}_y = R_y M_{ye} & , \ \tilde{b}_y = b_y M_{ye} & , & \tilde{d}_{yy} = d_{yy} M_{ye} & , & \tilde{d}_{yx} = d_{yx} M_{ye} & , & \tilde{b}_{yx} = b_{yx} M_{ye}
\end{array}\right\} \text{5-1}$$

That is, we replace all the coefficients with tildes over them by untilded coefficients multiplied times the corresponding (albeit unknown) equilibrium value for the state variable in whose evolution equation the coefficient appears. The new system of equations are given by:

$$\dot{I}_x = -I_x[\alpha_{xy} I_y + \gamma_{xy} M_y] - I_{xe}[\alpha_x I_x - C_{xx} M_x] + I_{xe} Q_x \qquad (5\text{-}2)$$

$$\dot{M}_x = -M_x[\delta_{xy} I_y + \beta_{xy} M_y] - M_{xe}[b_x M_x + d_{xx} I_x + b_{xy} M_y] + M_{xe} R_x \qquad (5\text{-}3)$$

$$\dot{I}_y = -I_y[\alpha_{yx} I_x + \gamma_{yx} M_x] - I_{ye}[a_y I_y - C_{yy} M_y] + I_{ye} Q_y \qquad (5\text{-}4)$$

$$\dot{M}_y = -M_y[\delta_{yx} I_x + \beta_{yx} M_x] - M_{ye}[b_y M_y + d_{yy} I_y + d_{yx} I_x + b_{yx} M_x] + M_{ye} R_y \qquad (5\text{-}5)$$

If we now equate the right hand sides to zero simultaneously when evaluated at $\{I_{xe}, M_{xe}, I_{ye}, M_{ye}\}$, we obtain the linear system of equations,

$$\left.\begin{array}{l}
\alpha_{xy} I_{ye} + \gamma_{xy} M_{ye} + a_x I_{xe} - c_{xx} M_{xe} - Q_x = 0 \\
(\delta_{xy} + d_{xy}) I_{ye} + (\beta_{xy} + b_{xy}) M_{ye} + d_{xx} I_{xe} + b_x M_{xe} - R_x = 0 \\
\alpha_{yx} I_{xe} + \gamma_{yx} M_{xe} + a_y I_{ye} - c_{yy} M_{ye} - Q_y = 0 \\
(\delta_{yx} + d_{yx}) I_{ye} + (\beta_{yx} + b_{yx}) M_{xe} + d_{yy} I_{ye} + b_y M_{ye} - R_y = 0
\end{array}\right\} (5\text{-}6)$$

which we may collect into matrix form as,

$$\begin{bmatrix} a_x & -c_{xx} & \alpha_{xy} & \gamma_{xy} \\ d_{xx} & b_x & (\delta_{xy}+d_{xy}) & (\beta_{yx}+b_{xy}) \\ \alpha_{yx} & \gamma_{yx} & a_y & -c_{yy} \\ (\delta_{yx}+d_{yx}) & (\beta_{yx}+b_{yx}) & & b_y \end{bmatrix} \begin{bmatrix} I_{xe} \\ M_{xe} \\ I_{ye} \\ M_{ye} \end{bmatrix} - \begin{bmatrix} Q_x \\ R_x \\ Q_y \\ R_y \end{bmatrix} = \underset{\sim}{0} \quad (5\text{-}7)$$

The system (5-7) may be solved for the equilibrium values given all the untilded coefficients, then the tilded coefficients of (5-1) can be computed and reinserted into the original system of equations since the equilibrium values are now known. This is a bootstrap method to find one equilibrium point. The location of the others remains unknown.

Probably the most important application of (5-7) is the means it provides us to establish an equilibrium point at an arbitrary location in the state space, by adjusting 4 of the parameters. In particular, we can easily calculate the resupply rates that are needed to establish a stationary point for the conflict in a region of the state space that is physically accessible. From an analytical point of view, establishing equilibrium at $\{1, 1, 1, 1\}$ is particularly convenient. If we do this, then we have the equations

$$\begin{aligned} a_x - c_{xx} + \alpha_{xy} + \gamma_{xy} &= Q_x \\ d_{xx} + b_x + d_{xy} + \delta_{xy} + b_{xy} + \beta_{xy} &= R_x \\ a_y - c_{yy} + \alpha_{yx} + \gamma_{yx} &= Q_y \\ d_{yy} + b_y + d_{yx} + \delta_{yx} + b_{yx} + \beta_{yx} &= R_y \end{aligned} \qquad (5\text{-}8)$$

that must be satisfied. These relationships only apply at unity equilibrium. Equation (5-7) must be used for the general case but it is already solved for the resupply vector, given the remaining parameters and any desired equilibrium point.

Once an equilibrium point is found, or the resupply rates are found to establish a specified point, we know that the rates of changes are simultaneously zero at that point and if the conflict somehow reaches it, the system state variables will, in theory, change no more. However, we are interested in discovering whether the point is stable or not. Is it like considering a marble either in the bottom of a bowl, or on top of an upside-down bowl. Or we may think of a pencil hanging from one end, or standing straight up on an end. In one case, a slight tap and the pencil falls over, in the other it returns to its original vertical position.

Similarly, the marble given a slight displacement rolls off the bowl in one case, but returns to the bottom, its equilibrium position, in the other. One type of equilibrium point is said to have "neighborhood instability", the other has "neighborhood stability". A system has "global stability" if it returns to equilibrium from anywhere in the space. Since we have a non-linear system, and there are multiple equilibria, we might expect, if we displace from the local neighborhood at a given point too far, that the system could return to a different point, much like valleys between different ranges of hills.

Thus our first task is to examine the "neighborhood stability" of the one equilibrium point we have found (or fixed). The standard technique for this is to introduce a small perturbation in the equilibrium state vector. The results are derived for the n^{th} order generalized Lotka-Volterra model in Appendix A. We repeat the derivation here for the four-species system Model V proposed in the previous section. Our mathematical model of system evolution is of the general form

$$\dot{\underset{\sim}{S}} = \underset{\sim}{F}(\underset{\sim}{S}, \underset{\sim}{\mu}) + \underset{\sim}{\tilde{Q}} \tag{5-9}$$

where specifically, $\underset{\sim}{S} = [I_x\ M_x\ I_y\ M_y]^T$, and the quadratic vector functions are given in equations (4-1). The resupply vector,

$\underset{\sim}{\tilde{Q}} = [\tilde{R}_x\ \tilde{Q}_x\ \tilde{R}_y\ \tilde{Q}_y]^T$, is presumed to be constant. We now define the perturbation vector $\underset{\sim}{s} = [i_x\ m_x\ i_y\ m_y]^T$ as the distance from the equilibrium point, $\underset{\sim}{S}_e = [I_{xe}\ M_{xe}\ I_{ye}\ M_{ye}]^T$ to the point $\underset{\sim}{S}$. Note that $\dot{\underset{\sim}{S}} = \dot{\underset{\sim}{s}}$ since $\underset{\sim}{S}_e$ is a constant. Now (5-9) becomes

$$\dot{\underset{\sim}{s}} = \underset{\sim}{F}\,(\underset{\sim}{S}_e + \underset{\sim}{s}, \underset{\sim}{\mu}) + \underset{\sim}{\tilde{Q}} \tag{5-10}$$

If we expand the non-linear functions about $\underset{\sim}{S}_e$ in a Taylor series and assume terms of quadratic degree and greater in $\underset{\sim}{s}$ are negligible for small perturbations, then

$$\dot{\underset{\sim}{s}} = \underset{\sim}{F}\,(\underset{\sim}{S}_e) + \underset{\sim}{C} \cdot \underset{\sim}{s} + \underset{\sim}{\tilde{Q}} \tag{5-11}$$

where $\underset{\sim}{C}$ is a four by four matrix with elements

$$c_{ij} = \left. \frac{\partial F_i}{\partial S_j} \right|_{\underset{\sim}{S} = \underset{\sim}{S}_e} \tag{5-12}$$

Now since $\underset{\sim}{S}_e$ is an equilibrium point, then $F\,(\underset{\sim}{S}_e) + \underset{\sim}{Q} = \underset{\sim}{0}$ by definition. Therefore

$$\dot{\underset{\sim}{s}} = \underset{\sim}{C} \cdot \underset{\sim}{s} \tag{5-13}$$

are the linear homogenous state equations that govern the dynamical behavior of the perturbation variables $\underset{\sim}{s}$ near the equilibrium point $\underset{\sim}{S}_e$. The matrix $\underset{\sim}{C}$ couples those variables together and, as is explained briefly in Appendix A, it is well known from the theory of linear dynamical systems that its eigenvalues determine the stability of the equilibrium point. We shall call the $\underset{\sim}{C}$ that goes with our four species model of modern military conflict the "Conflict Matrix".

Since the specific terms retained in the model in this paper identify it as Model V, we shall designate the corresponding matrix as "Conflict Matrix V", or $\underset{\sim}{C}_v$.

The elements of $\underset{\sim}{C}_v$, found according to Eqn (5-12) from Eqn (4-1) yield,

$$\underset{\sim}{C}_v = \begin{bmatrix} (-a_x I_{xe} - \alpha_{xy} I_{ye} - \gamma_{xy} M_{ye}) & c_{xx} I_{xe} & -\alpha_{xy} I_{xe} & -\gamma_{xy} I_{xe} \\ -d_{xx} M_{xe} & (-b_x M_{xe} - \delta_{xy} I_{ye} - \beta_{xy} M_{ye}) & (-\delta_{xy} - d_{xy}) M_{xe} & (-\beta_{xy} - b_{xy}) M_{xe} \\ -\alpha_{yx} I_{ye} & -\gamma_Y + I_{ye} & (-a_y I_{ye} - \alpha_{yx} I_{xe} - \gamma_{yx} M_{xe}) & c_{yy} I_{ye} \\ (-\delta_{yx} - d_{yx}) M_{ye} & (-\beta_{yx} - b_{yx}) M_{ye} & -d_{yy} M_{ye} & (-b_y M_{ye} - \delta_{yx} I_{xe} - \beta_{yx} M_{xe}) \end{bmatrix} \quad (5\text{-}14)$$

Note that the elements in the conflict matrix, and hence its eigenvalues, depend on the equilibrium point as well as the system parameters. For analytical purposes, fixing the equilibrium point at unity, $\underset{\sim}{S}_e = [1\ 1\ 1\ 1]^t$, establishes the somewhat simpler conflict matrix

$$\underset{\sim}{C}_v = \begin{bmatrix} -(a_x + \alpha_{xy} + \gamma_{xy}) & c_{xx} & -\alpha_{xy} & -\gamma_{xy} \\ -d_{xx} & -(b_x + \delta_{xy} + \beta_{xy}) & -(d_{xy} + \delta_{xy}) & -(b_{xy} + \beta_{xy}) \\ -\alpha_{yx} & -\gamma_{yx} & -(a_y + \alpha_{yx} + \gamma_{yx}) & c_{yy} \\ -(d_{yx} + \delta_{yx}) & -(b_{yx} + \beta_{yx}) & -d_{yy} & -(b_y + \delta_{yx} + \beta_{yx}) \end{bmatrix} \quad (5\text{-}15)$$

The solution of the state Eqn (5-13) is of the form

$$s_i(t) = \sum_{j=1}^{4} r_{ij}\, e^{P_j t} \quad , \quad i=1,2,3,4 \quad , \quad t \geq 0 \quad (5\text{-}16)$$

The $\{P_j\}$, eigenvalues of $\underset{\sim}{C}$, are the roots of the characteristic eqn,

$$D(p) = \mathrm{Det}[p\underset{\sim}{I} - \underset{\sim}{C}]$$

For our four-species Model V, $D(p)$ is a fourth order polynomial with coefficients that depend, in a very complicated way, on the elements of $\underset{\sim}{C}_v$.

The $\{r_{ij}\}$ are constants that depend on the initial values of the perturbation vector, $\underset{\sim}{s}(o)$.

The important thing, however, are the roots $\{p_j\}$, which will either cause the perturbation to die out, if all the Re $\{p_j\} < 0$, in which case we have "neighborhood stability", or cause the perturbation to grow if Re $\{p_j\} > 0$ for one or more roots. In this latter case we say the system is "unstable at $\underset{\sim}{S} = \underset{\sim}{S}_e$". (It may be stable at another equilibrium point).

In the four species model there are four roots or eigenvalues. They may all be real, there may be two real and one complex conjugate pair, or there may be two complex conjugate pairs. The system is stable for a particular combination of coefficients if all the roots lie in the <u>R</u>ight <u>H</u>alf of the complex plane (RHP). Of course, if any one (or more) of the model coefficients is varied, the roots will move about in the complex plane. If a smooth variation of the coefficients causes one or more roots to move into the Left Half Plane (LHP), the system is said to be "environmentally unstable".

It is very important to determine if our model exhibits, or is capable of exhibiting this type of instability. What it suggests is the possibility of a stalemated (stable) battle situation changing to a battle with a decisive outcome (one side being severely depleted or wiped out) as the result of only a small change in one of the model coefficients, e.g. C^3 effectiveness (d_{xy} or d_{yx}). In fact it is possible, that in an effort to win the battle (or war), the C^3 or firepower effectiveness might be increased, the equilibrium point becomes unstable, but due to chance the opposition may gain a temporary slight advantage and he is in fact able to win. We shall see that this phenomenon, known as a "bifurcation", can indeed occur in our system as illustrated by the example described in the next section.

VI. C^3 Effectiveness and Environmental Instability

As an illustration of dynamics of Conflict Model V, we investigate the effects of variation in the linear, C^3-effectiveness coefficients d_{xy} (effectiveness of Y against X) and d_{yx} (effectiveness of X against Y), while holding all the other coefficients constant. Table 6.1 lists the values we have selected, somewhat arbitrarily, for all the other parameters. We might note the following from Table 6.1:

1.) Only Lanchester "area fire" terms are retained. "Aimed fire" is set to zero.

2.) Quadratic C^3-effectiveness terms are set to zero.

3.) The example is symmetrical except for the variable terms, d_{xy} and d_{yx}, which may or may not be equal, and for the counter-C^3 terms. Note that X employs pure physical destruction of Y's C^3 assets, with no deception ($\gamma_{yx} \neq 0$, $\alpha_{yx} = 0.0$) whereas Y employs pure deception and misinformation against X, with no physical destruction ($\alpha_{xy} \neq 0$, $\gamma_{xy} = 0.0$). This example is asymmetrical in counter-C^3.

Let us now calculate the resupply rates in order to establish an equilibrium point at unity; $\{I_{xe} = 1.0, M_{xe} = 1.0, I_{ye} = 1.0, M_{ye} = 1.0\}$. According to Eqns (5-8),

$$
\begin{aligned}
Q_x &= 1.5 \\
R_x &= 1.5 + d_{xy} \\
Q_y &= 1.5 \\
R_y &= 1.5 + d_{yx}
\end{aligned}
\qquad (6\text{-}1)
$$

gives the rates of resupply necessary to maintain steady state force and information levels of 1.0 for both sides. Note that the force resupply rates depend on the C^3 effectiveness coefficients directly. For example, if side Y can increase the effectiveness of his forces against X thru

Table 6.1

Model V Parameters for Example Calculations
(Variable C^3 Effectiveness)

Natural Loss Rate Coefficients

$a_x = .5$ $\quad$ $a_y = .5$

$b_x = .5$ $\quad$ $b_y = .5$

Diversion of Resources to C^3

$c_{xx} = 0.0$ $\quad$ $d_{xx} = 0.0$ $\quad$ $c_{yy} = 0.0$ $\quad$ $d_{yy} = 0.0$

Lanchester Fire Effectiveness Coefficients

$b_{xy} = 0.0$ $\quad$ $b_{yx} = 0.0$ $\quad$ $\beta_{xy} = 1.0$ $\quad$ $\beta_{yx} = 1.0$

C^3 Effectiveness Coefficients

d_{xy} = variable parameter $\quad$ d_{yx} = variable parameter $\quad$ $\delta_{xy} = 0.0$ $\quad$ $\delta_{yx} = 0.0$

Counter C^3 Coefficients

Physical Destruction

$\gamma_{xy} = 0.0$ $\quad$ $\gamma_{yx} = 1.0$

Deception and Misinformation

$\alpha_{xy} = 1.0$ $\quad$ $\alpha_{yx} = 0.0$

improved C^3, that is Y is able to increase d_{xy}, then X must resupply his forces at a greater rate, R_x increases, in order to maintain a constant level of men and material in the field. This seems reasonable. Note that each side is supplying information at a rate sufficient to make up for the natural decay rate of information (.5), plus the losses of information due to the opposition's counter-C^3 activities.

The complete non-linear dynamical equations for this example are, from Eqns (4-1)

$$\begin{aligned}
\dot{I}_x &= 1.5 - I_x(I_y + 0.5) \\
\dot{M}_x &= 1.5 + d_{xy} - M_x(M_y+0.5) - d_{xy}I_y \qquad (6\text{-}2) \\
\dot{I}_y &= 1.5 - I_y(M_x + 0.5) \\
\dot{M}_y &= 1.5 + d_{yx} - M_y(M_x + 0.5) - d_{yx}I_x
\end{aligned}$$

In this particular example, because of the large number of zero parameters in Table 6.1, there are only four equilibrium points, one of which is unity. (That unity is an equilibrium point of (6-2) is easily checked by substituting one simultaneously for I_x, M_x, I_y, and M_y and noting that all four equations are zero.)

It is possible to study numerically the behavior of this system of equations quite simply by using discreet time methods and a computer. However, we can focus this study more effectively by analyzing the stability of the system near equilibrium as a function of d_{xy} and d_{yx}. For this we need the conflict matrix and its eigenvalues. Substituting our parameters from Table 6.1 into Eqn (5-15), the conflict matrix for unity equilibrium is:

$$\underset{\sim}{C}_v = \begin{bmatrix} -1.5 & 0 & -1.0 & 0 \\ 0 & -1.5 & -d_{xy} & -1.0 \\ 0 & -1.0 & -1.5 & 0 \\ -d_{yx} & -1.0 & 0 & -1.5 \end{bmatrix} \qquad (6\text{-}3)$$

We have investigated the eigenvalues of this matrix for the following two cases:

a.) d_{xy} held fixed at 1.0 and d_{yx} varied from 0.0 to 2.0

b.) d_{yx} held constant at 1.0 and d_{xy} varied from 0.0 to 2.0.

The IMSL routine EIGRF was employed to calculate the eigenvalues numerically [10].

The results of these calculations can be illustrated graphically by plotting the locus of the eigenvalues of $\underset{\sim}{C}_v$ in the complex plane for the two cases described above. Figure 6.1 a.) shows the loci of the roots for case a.) and Fig. 6.1 b.) shows the loci for case b.). Note in both cases there are two real roots and one complex conjugate pair. The complex pair moves along a constant abscissa of -1.5 as C^3 effectiveness is varied. These roots lead to damped oscillatory terms in the system response and are stable.

In each case, both of the real roots begin in the left half plane, but one moves left and the other moves right as C^3 effectiveness is increased. In both cases the one moving right winds up in the RHP to become an unstable, or exponentially growing term in the response.

In case a.), the root crosses the imaginary axis for $d_{yx} = \hat{d}_{yx} = 0.56$ and in case b.) for $d_{xy} = \hat{d}_{xy} = 0.81$ Thus in case a.), for $d_{yx} > .56$

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(10) "Eigenvalues and (optionally) Eigenvectors of a Real General Matrix in Full Storage Mode", Copyright by IMSL Inc., 1978.

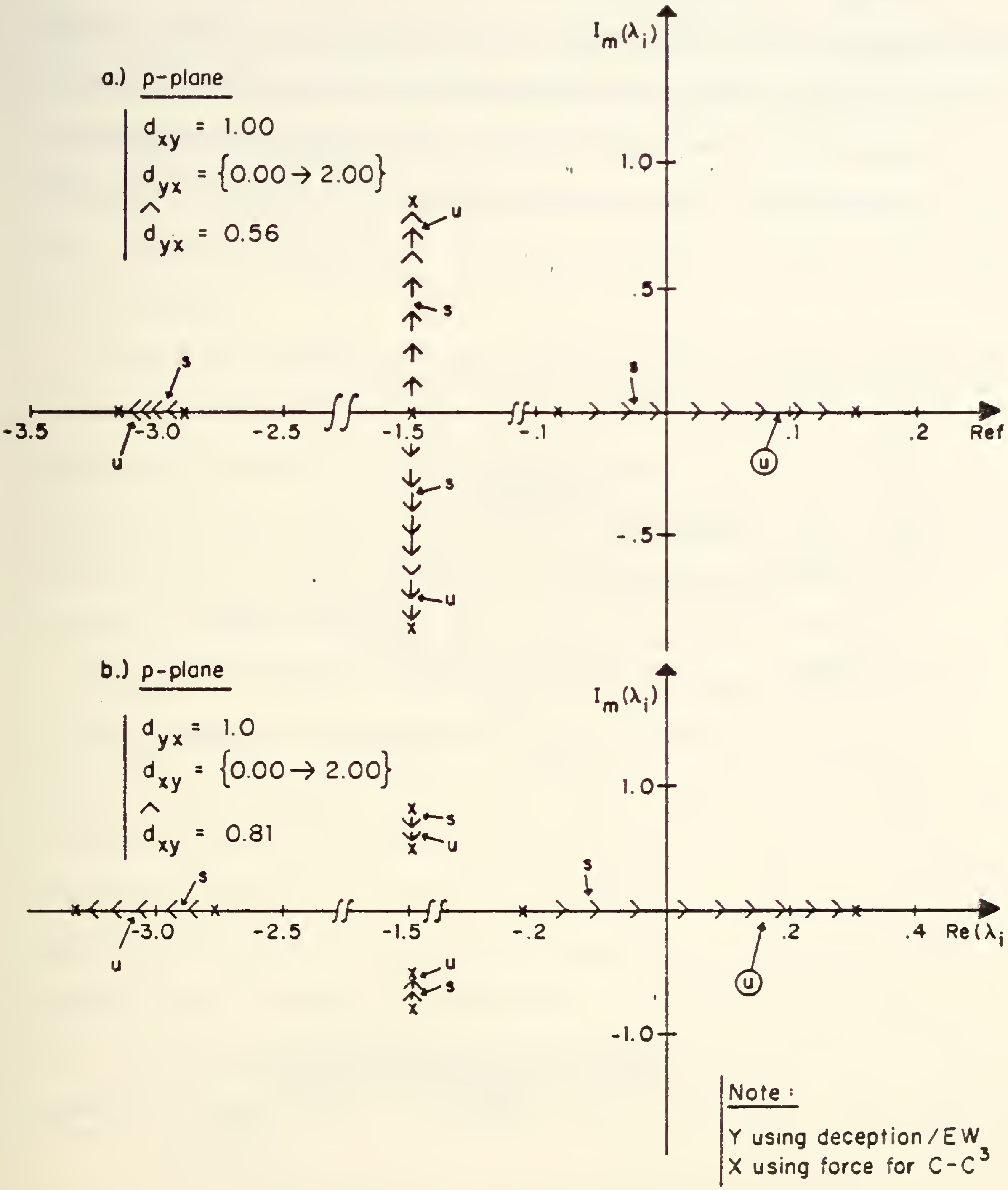

Figure 6.1. Root Locus Plots for C_v (Eigenvalues).

the system is unstable. In case b.), for $d_{xy} > .81$ the system is unstable. The cases are not symmetrical because of the asymmetrical counter-C^3 policies of the two sides.

It is quite straightforward to determine the curve in the $\{\{d_{xy}, d_{yx}\}$ parameter space that bisects that space into stable and unstable regions by examining the characteristic polynomial for the $\underset{\sim}{C}_v$ given in Eqn (6-3). The result is shown in Figure 6-2. (The bisecting curve happens to be a straight line.) The two critical values from Figure 6.1 are marked by X's.

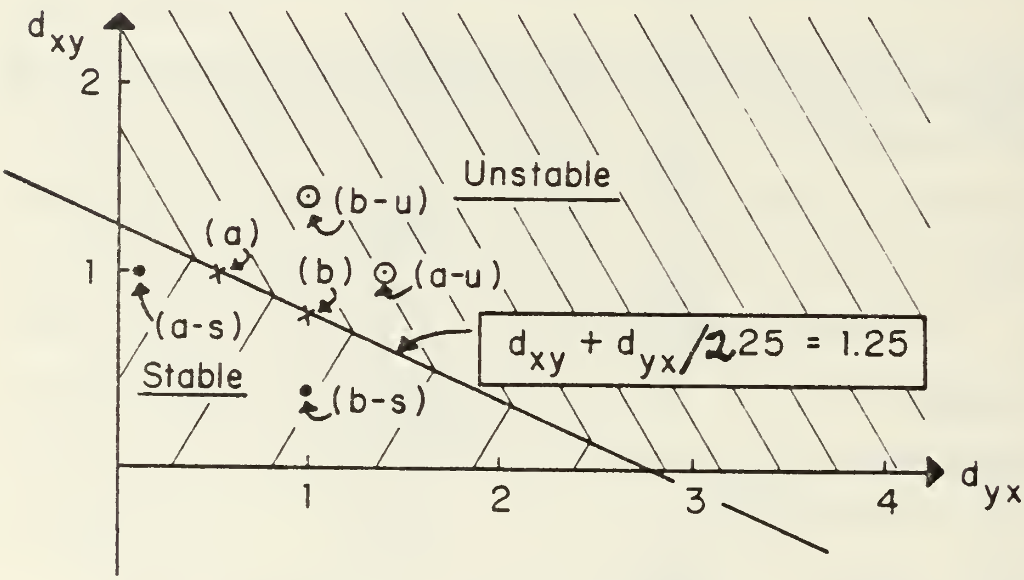

Figure 6.2

Stable and Unstable Regions of C3I Effectiveness Parameter Space

Returning to Figure 6.1 a.) and b.), we note sets of eigenvalues marked "u" for unstable and "s" for stable. These points are also indicated in the parameter diagram of Figure 6.2. These four points in Figure 6.2 have been investigated numerically for small perturbations from system equilibrium using the full system of nonlinear equations (6-2). The Interactive Ordinary Differential Equations (IODE) package [11] and the NPS IBM 3033 computer were utilized in these studies. We present here some of our results for Case b.) for both the stable and unstable points marked in Figure 6.2 as "(b-s)" and "(b-u)". The results for Case a.) are similar.

A number of different initial conditions were investigated and these are listed in Table 6.2 and Table 6.3. In the stable case, Table 6.2, the system always returns to its equilibrium value, {1.0, 1.0, 1.0, 1.0}, although the time constants to damp out the perturbation vary, depending on the direction of the perturbation, from as small as one time unit to as great as 17 time units.

Figure 6.3 shows the time history of all four variables and Figure 6.4 shows the force level phase plot, M_y vs M_x, for Trial #2, a stable case with side X given on a 25% initial force advantage. This advantage is wiped out after approximately seven time units and the forces become balanced at equilibrium. Note Side Y's initial loss in force is counter-balanced by his superior rate of replenishment. (R_y = 2.5, R_x, 1.9). However, side X's superior C^3 effectiveness (1.0 vs 0.4) enables him to hold Y at a stalemate in steady state with 75% the rate of force replenishment. It seems natural to describe side X's superior c^3 effectiveness

(11) Hillary, R.R., "Interactive Ordinary Differential Equations Package", W.R. Church Computer Center, User's Guide to WM/CMS at NPS, April 1981.

Table 6.2

Initial Conditions for Case b.) with d_{xy} at 0.4 and d_{yx} at 1.0 (Stable)

Trial #1	Time	Increment	I_x	M_x	I_y	M_y	Time for 90% Damp Out (time units)
1	0 to 50	.05	1.25	1.00	1.00	1.00	7
2	0 to 50	.05	1.00	1.25	1.00	1.00	12
3	0 to 50	.05	1.00	1.00	1.25	1.00	9
4	0 to 50	.05	1.00	1.00	1.00	1.25	10
5	0 to 50	.05	1.25	1.25	1.00	1.00	17
6	0 to 50	.05	1.25	1.00	1.25	1.00	1
7	0 to 50	.05	1.25	1.00	1.00	1.25	1.5
8	0 to 50	.05	1.00	1.25	1.00	1.25	1.5
9	0 to 50	.05	1.00	1.00	1.25	1.25	15.5
10	0 to 50	.05	1.00	1.25	1.25	1.00	4.0
11	0 to 50	.05	1.25	1.25	1.25	1.00	12.0
12	0 to 50	.05	1.25	1.25	1.00	1.25	11.0
13	0 to 50	.05	1.25	1.00	1.25	1.25	12.0
14	0 to 50	.05	1.00	1.25	1.25	1.25	4.0

Table 6.3

Initial Conditions for Case b.) with d_{xy} at 1.4 and d_{yx} at 1.0 (Unstable)

Trial #1	Time	Increment	I_x	M_x	I_y	M_y	Increasing Force Term
1	0 to 50	.05	1.25	1.00	1.00	1.00	M_x
2	0 to 50	.05	1.00	1.25	1.00	1.00	M_x
3	0 to 50	.05	1.00	1.00	1.001	1.00	M_x
4	0 to 50	.05	1.00	1.00	1.00	1.001	M_y
5	0 to 50	.05	1.25	1.25	1.00	1.00	M_x
6	0 to 50	.05	1.001	1.00	1.001	1.00	M_y
7	0 to 50	.05	1.001	1.00	1.00	1.001	M_y
8	0 to 50	.05	1.00	1.00	1.001	1.001	M_y
9	0 to 50	.05	1.00	1.001	1.00	1.001	M_x
10	0 to 50	.05	1.00	1.01	1.01	1.00	M_y
11	0 to 50	.05	1.25	1.25	1.25	1.00	M_x
12	0 to 50	.05	1.25	1.25	1.00	1.25	M_x
13	0 to 50	.05	1.001	1.000	1.001	1.001	M_y
14	0 to 50	.05	1.00	1.001	1.001	1.001	M_y

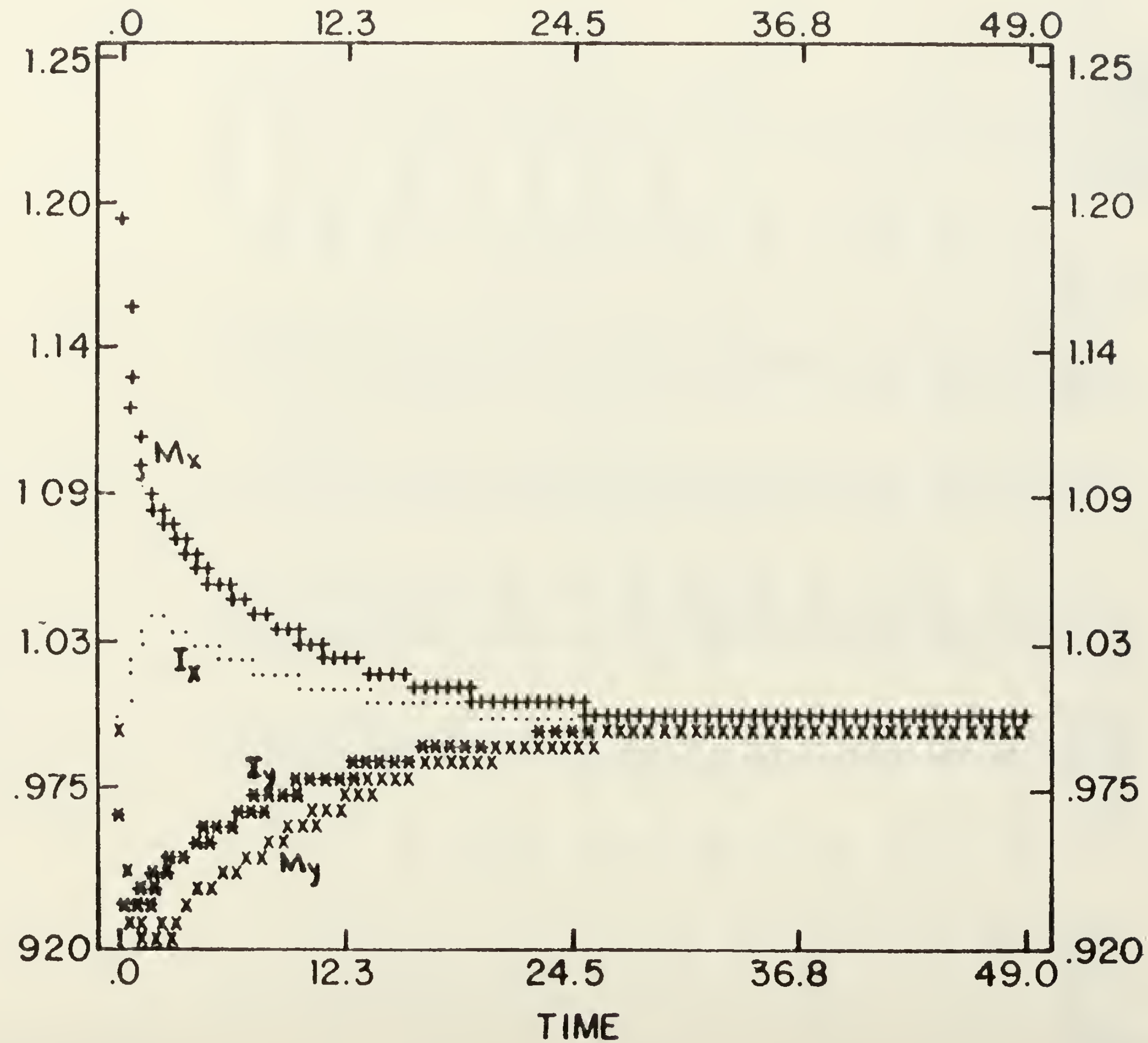

Figure 6.3. Single Perturbation Stable (M_X = 1.25).

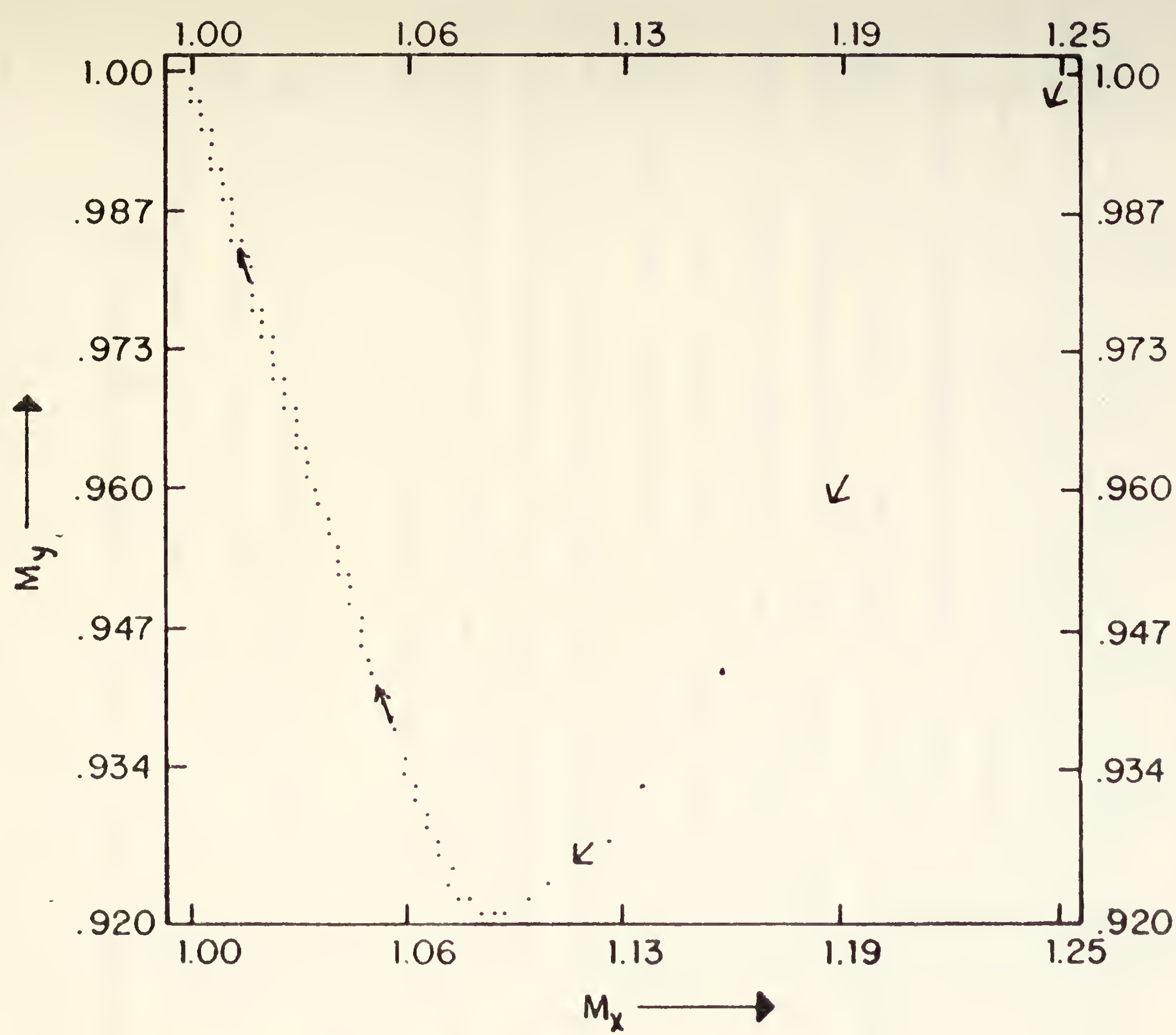

Figure 6.4. Single Perturbation Stable Phase Plot (M_X at 1.25).

as giving him a "force multiplication factor" of 4/3. In general, we shall define

$$\Gamma_{yx}(\hat{\underset{\sim}{\mu}}) = \frac{R_x(\hat{\underset{\sim}{\mu}})}{R_y(\hat{\underset{\sim}{\mu}})} \tag{6-4}$$

as $X^{'s}$ force multiplication factor versus Y where $\hat{\underset{\sim}{\mu}}$ is the surface in the parameter space $\hat{\underset{\sim}{\mu}}$ that produces a stable equilibrium point at unity.

Figure 6.5 corresponds to Trial #2 in Table 6.2. Here the phase plot illustrates how a temporary increase in knowledge for side X gives X a temporary gain in forces of about 4% and Y a temporary decrease in force level of about 7%. However, due to the system parameters, $X^{'s}$ advantage is in fact only temporary and the natural dynamic relationships between the variables of the system drive it back to equilibrium.

Figure 6.6 corresponds to Trial #6 of Table 6.2. Here both X and Y have initial information levels greater than their equilibrium values by 25%. The phase plot shows how initially both forces levels decrease, then Y appears to be gaining an advantage over X (Y is using deception), but slowly closes the gap and the force levels return to equilibrium ($M_{xe} = M_{ye} = 1.0$).

It must be emphasized that the dynamics shown are those that occur if neither X nor Y change any of their policies, doctrines or efforts (parameters are held constant). One might, if he sees his force dimishing, attempt to change his fortunes by doctrinal or policy or motivational improvement.

For example, Y, seeing that he is losing, as in the initial phases of Trial #2, Figure 6.5, might be able to increase his C^3 effectiveness from $d_{xy} = 0.4$ to $d_{xy} = 1.4$. It would now seem he would be at an advantage since $d_{yx} = 1.0$ is $X^{'s}$ C^3 effectiveness. However, $Y^{'s}$

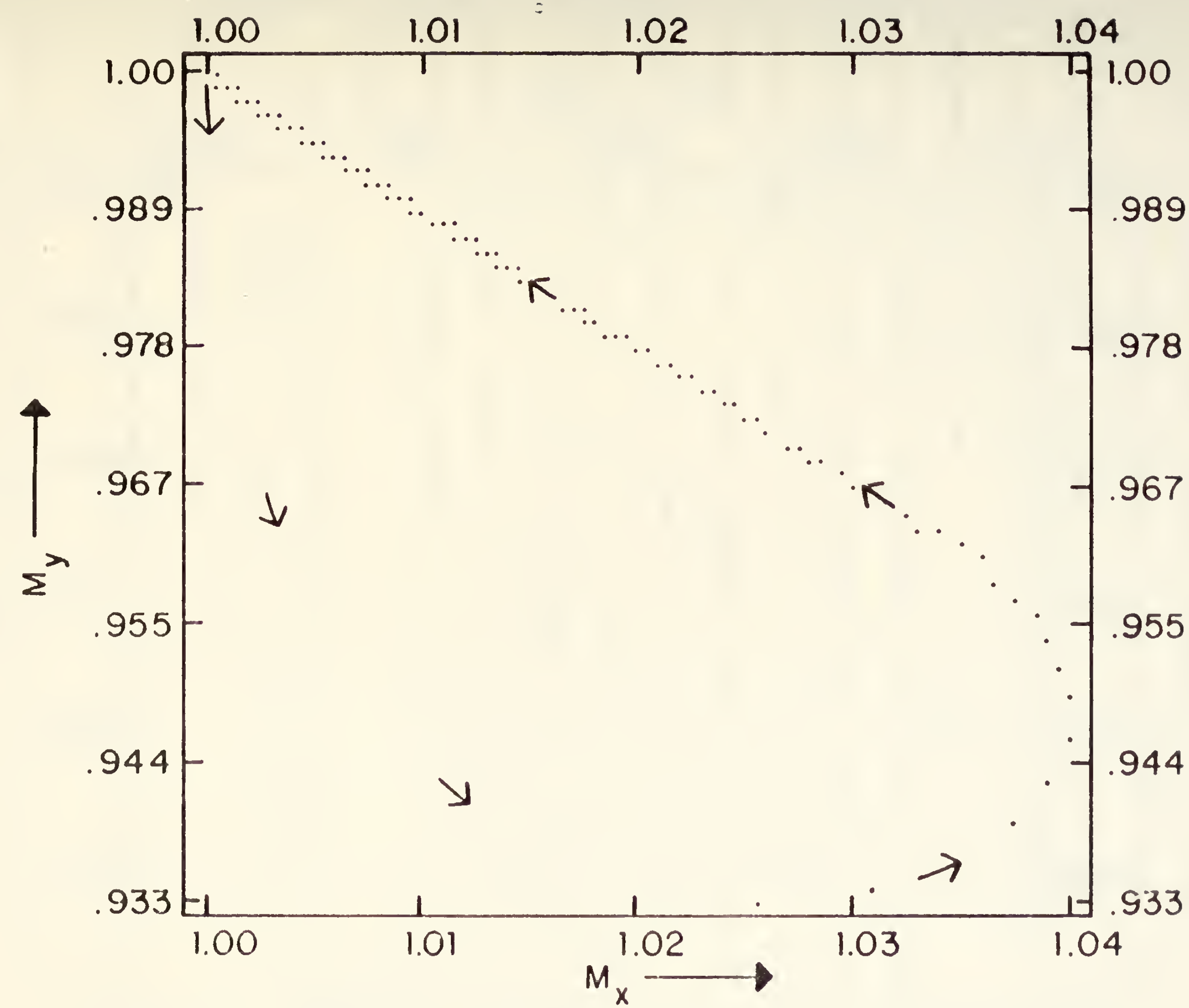

Figure 6.5. Single Perturbation Stable Phase Plot (I_x at 1.25).

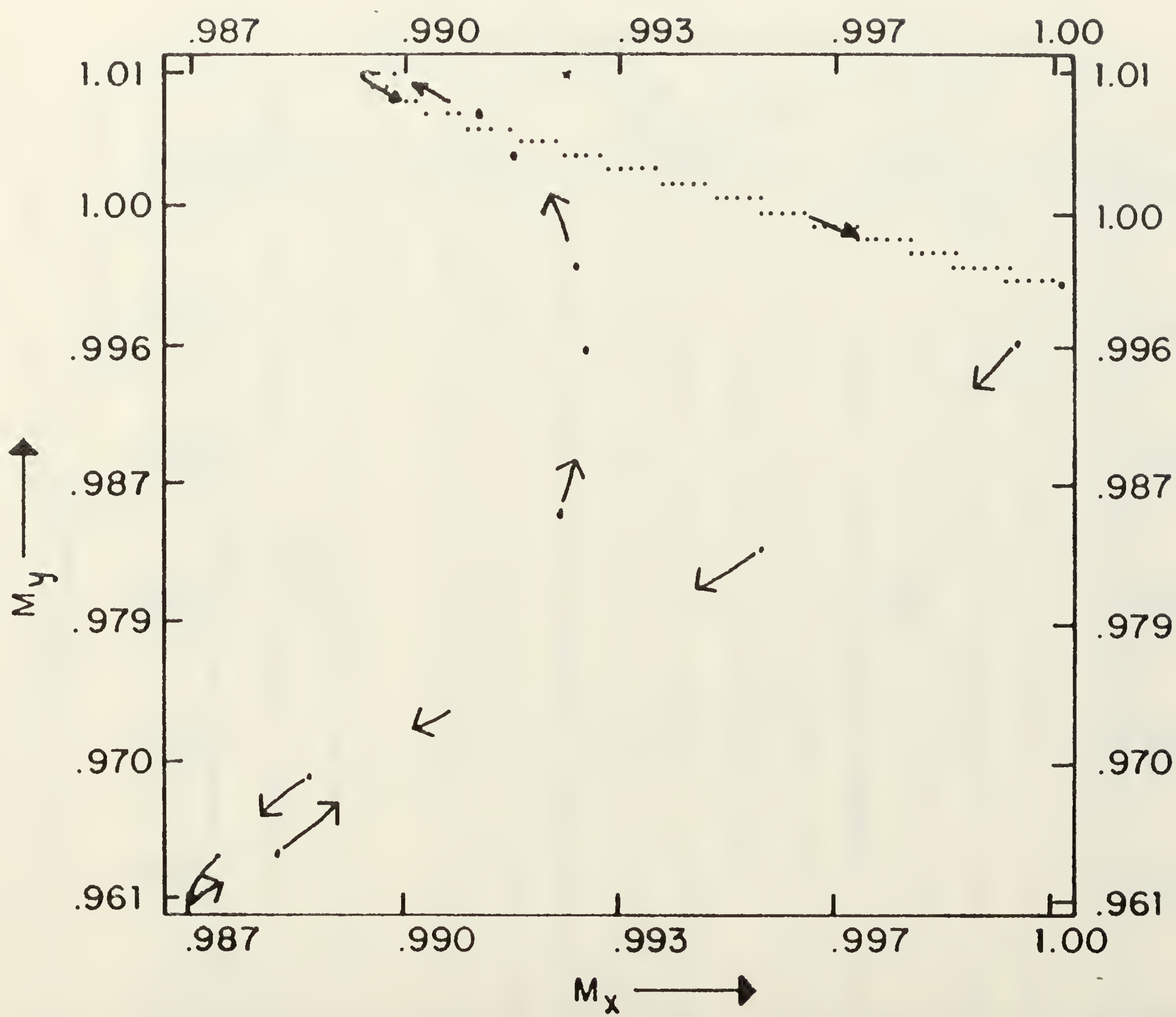

Figure 6.6. Double Perturbation Stable Phase Plot (I_x and I_y at 1.25).

parameter variation has altered the system to an unstable mode and one side or the other will tend to win. Unfortunately for Y, if X is initially at a force advantage, X will tend to win, as is shown in Figures 6.7 and 6.8. These figures corresponds to Trial #2 of Table 6.3, the table of numerical trials for the unstable case "(b-u)" in Figures 6.1 and 6.2. The other trials of Table 6.3 do show that if Y can gain an advantage along with his doctrinal or procedural improvement he can in fact win!

The behavior exhibited by this system is indicative of "environmentally unstable" systems, systems where relatively small parameter changes cause dramatic changes in the dynamic behavior. Furthermore, the fact that when unstable, either X or Y may win, depending upon who achieves the first slight advantage, which may of course be by chance, even though Y has the most effective force ($d_{xy} > d_{yx}$), is illustrative of what is called a bifurcation. If only Y can win in the unstable case, regardless of which side gains the initial advantage, there is no bifurcation. It is obviously important to know whether an unstable equilibrium point exhibits this property.

Finally, Figure 6.7 demonstrates another very interesting phenomenon. The unstable system is only unstable at the unity equilibrium point. In fact, there is another equilibrium point at

$$\{M_{xe'} = 3.206,\ I_{xy'} = 1.647,\ M_{ye'} = .228,\ I_{ye'} = .405\}$$

which is stable and the system comes to steady state at this point. Although X has a distinct force and information advantage over Y, Y is resupplying both men and materials and intelligence fast enough that X is unable to completely wipe him out.

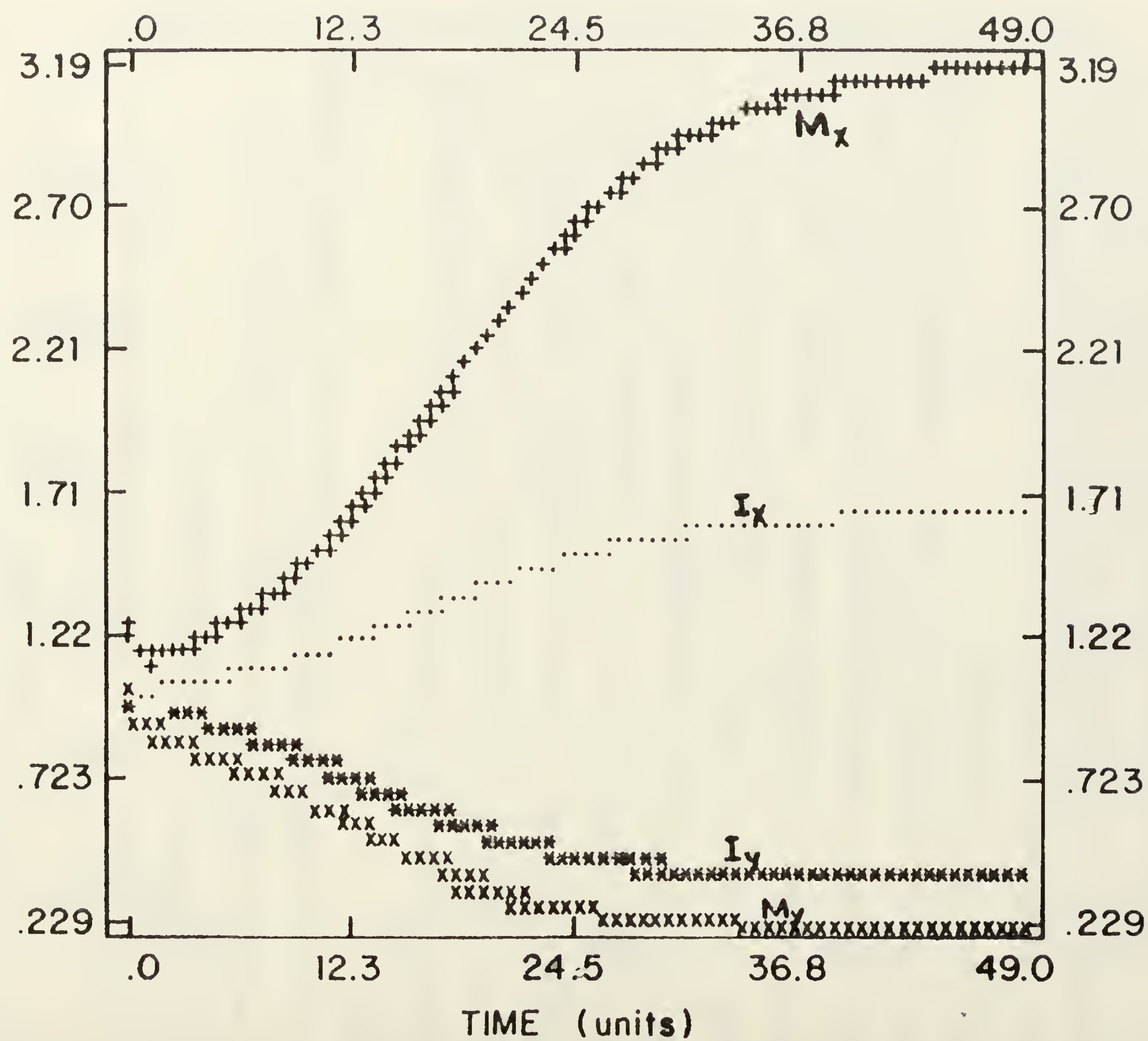

Figure 6.7. Single Perturbation Unstable (M_x at 1.25).

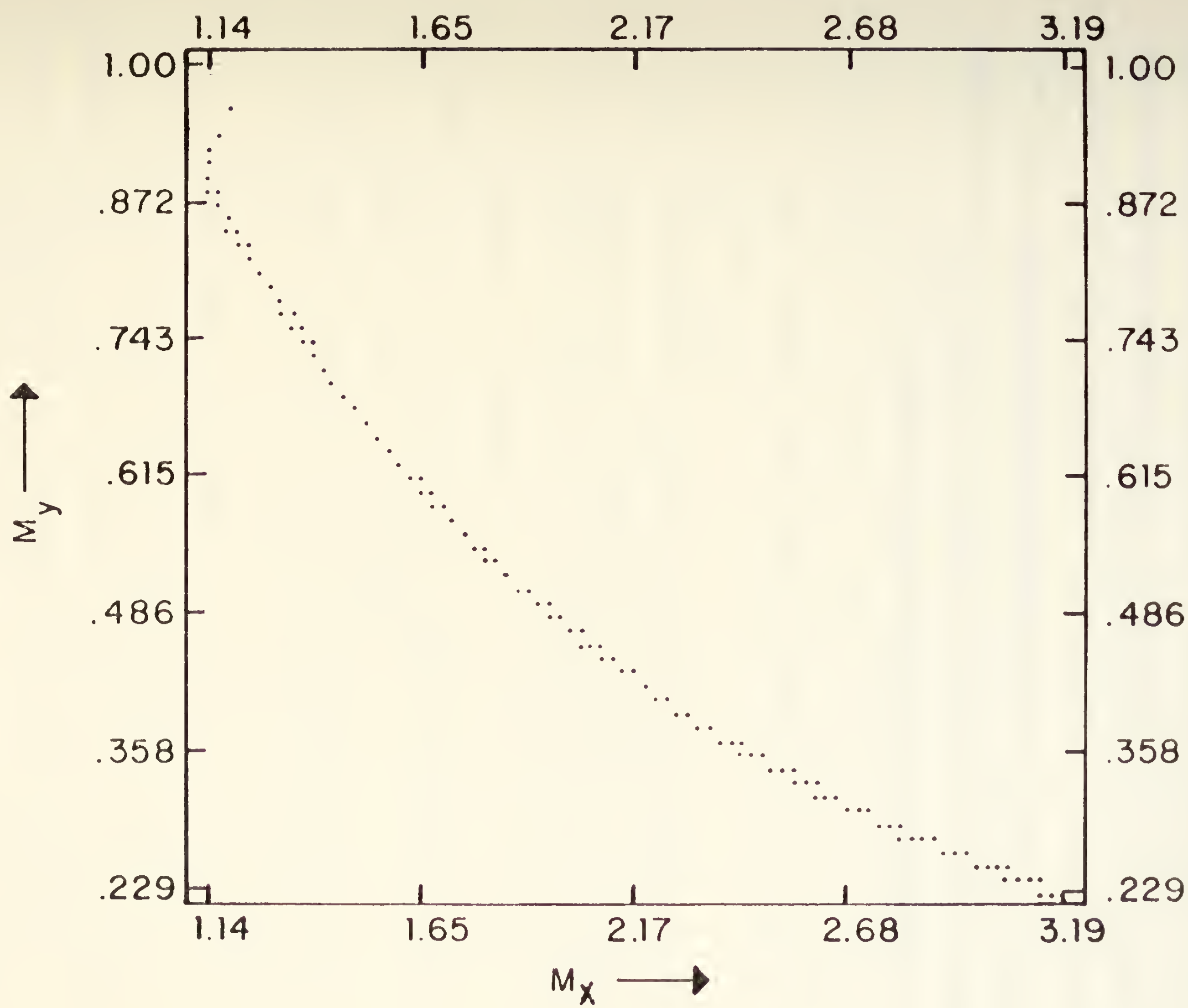

Figure 6.8. Single Perturbation Unstable Phase Plot (M_x at 1.25).

In the example described in this section, there are as we have previously pointed out, four equilibrium points. We have found two, one by design at unity and the other by chance through numerical trials.

One of these two is unstable but the second is stable and the system will gravitate to the stable point. It is important to note that both of these points are in the physically realizable region of the four-dimensional space of our four species. All species must be real, non-negative numbers to be physically realized. The locations of the other two equilibria for this example are still unknown.

VII. Discussion

The problem of accounting for the importance of C^3 and intelligence information on the outcome of military conflicts - individual battles or entire campaigns - has been attacked in this paper by mathematical modeling with evolution equations. The four species model we have proposed here has the following major benefits:

1. It predicts both dynamic and average or steady state force levels.
2. It is non-linear, which appears imperative in order to account for the relationships between forces and information.
3. Because it is non-linear, there are multiple possible steady state solutions, some of which may be stable and some of which may be unstable.
4. The parameters of the model can be directly associated with C^3I features such as C^3 effectiveness, counter-C^3 (including deception, jamming, spoofing, etc.), intelligence information, and information "ageing" or "time late"
5. Force model includes ordinary aimed fire and area fire terms in addition to the C^3I related terms.
6. A natural expression occurs for the force multiplier of C^3I in terms of the ratio of rates of resupply required to maintain equal forces in the field.

The Volterra structure for evolution equations, which contains constant as well as linear and quadratic terms, was selected primarily because of its mathematical tractability (some major mathematical problems remain even with these equations). However, one can think of this model as containing the first two sets of terms in a power series expansion of the true attrition functions, whatever they may be. This type of argument has been made in the ecosystem modeling literature [May, op. sit.]. Whether it is valid or not for conflict modeling needs more discussion.

What can the model tell us? Assuming we can get the parameters somewhat nearly right, by analysis of data or through simulation, it can 1) tell us the possible steady state force levels and, 2) define the regions of the parameter space where they are stable and the regions where they are unstable. If a point is stable we can, 3) determine the characteristic time constants and oscillatory frequencies of the system when operating in the neighborhood of the stable point. If a point is unstable we can determine 4) what type of perturbations will cause one side or another to win (or in a stochastic model, the probability one side or the other will win) as well as the characteristic time constants for the system to diverge. Finally, for stable equilibrium points, we can 5) define the force multiplication due to improved intelligence, C^3 and counter-C^3.

It is clear that the model proposed here is capable of exhibiting a great variety of behaviors, behaviors not unlike those found in actual military conflicts. There are, however, clearly a great many issues that need additional thought and in some cases justification. First, there remain some important mathematical problems. First among these is the problem of locating the remaining equilibrium points and determining their stability. (Recall that only one can presently be determined analytically). There are several possible approaches to this problem including:

1.) Lyaponov Functions (Finding one for the model)

2.) Bifurcation Theory

3.) Continuation Methods (Numerical approach)

4.) A method of scaling similar to that used in Chap IV of this paper.

The second major mathematical problem relates to the extraordinary dependence of the eigenvalues of the Conflict Matrix, and hence system stability, on the linear terms in the equations. This needs further investigation and clarification. Put simply the quadratic terms are essential to the determination of the several possible steady state solutions, but the linear terms seem most important in determining their stability or lack thereof, i.e. whether or not the system can stay at a particular stationary point.

We also need to look at restructuring the model into a system of discrete time difference equations. This approach is still amenable to analysis. But it will more easily allow modeling of time delays in the supply of both forces and information and control. Numerical analysis is essentially unchanged.

Finally, we need to begin investigating means to test the model and to determine reasonable values for the parameters. Unless this can be done eventually, little ultimate gain will come from these efforts although we will certainly be able to gain some insights into the relative merits of various means of counter-C^3. Since it was the counter-C^3 problem that initiated these efforts originally, we shall pursue that application first.

Appendix A

The General Mathematical Theory

Systems

We have a system of N first order non-linear differential equations of the form

$$\underset{\sim}{S} = \underset{\sim}{F}\,(\underset{\sim}{S}) + \tilde{\underset{\sim}{Q}} \qquad \text{(A-1)}$$

This class of equations are known as evolution equations. $\underset{\sim}{S}$ and $\underset{\sim}{Q}$ are Nx1 column vectors and there are N functional relationships, which we shall take here to be at most quadratic. If we make each function of the form

$$F_i(S) = - S_i\left[\sum_{j=1}^{N} \alpha_{ij}\, S_i\right] - \sum_{j=1}^{N} \tilde{a}_{ij} S_j$$

the system is a very generalized version of the Lotka-Volterra equations for multi-species eco-systems. The system then, that we shall study is as in (A-1), but we write somewhat more explicitly that

$$S_i = - S_i\left(\sum_{j=1}^{N} \alpha_{ij}\, S_j\right) - \sum_{j=1}^{N} \tilde{a}_{ij}\, S_j + \tilde{Q}_i \quad , \quad i = 1,2...N \qquad \text{(A-2)}$$

Equilibrium Points

We are interested in the set of vectors, $\underset{\sim}{S}_e$ for which all the rates are zero, i.e., for which $\dot{\underset{\sim}{S}} = \underset{\sim}{0}$. Since the system is non-linear, there will be a number of these vectors, and the nonlinearity of the system makes their determination, in general, exceedingly difficult. However, there is a very ingenious way to determine one equilibrium vector from the solution of a linear system of equations. We rewrite equations (A-2) as follows:

$$S_i = - S_i\left(\sum_{j=1}^{N} \alpha_{ij}\, S_j\right) - \sum_{j=1}^{N} \alpha_{ij}\, S_{ie}\, S_j + Q_i\, S_{ie} \qquad \text{(A-3)}$$

Though we don't know S_{ie}, we know it is constant and that when $\dot{\underset{\sim}{S}} = \underset{\sim}{0}$, $\underset{\sim}{S} = \underset{\sim}{S}_e$.

Evaluating (A-3) at $\underset{\sim}{S} = \underset{\sim}{S}_e$ we find

$$\sum_{j=1}^{N} (\alpha_{ij} + a_{ij}) S_{je} = Q_i \quad , \quad \text{for} \quad i=1,2,..N \tag{A-4}$$

or in matrix notation

$$\underset{\sim}{Q} = \underset{\sim}{K}\, \underset{\sim}{S}_e \tag{A-5}$$

where the coefficient matrix is simply

$$\underset{\sim}{K} = \begin{bmatrix} \alpha_{11} & \alpha_{11} & \alpha_{1N} \\ \alpha_{21} & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \alpha_{N1} & & \alpha_{NN} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \cdot \cdot \cdot & a_{1N} \\ a_{21} & a_{22} \cdot \cdot \cdot & a_{2N} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ a_{N1} & a_{N2} \cdot \cdot \cdot & a_{NN} \end{bmatrix}$$

Quadratic Prediation (loss per capita per capita) — Linear Predation (loss per capita)

$$\underset{\sim}{K} = \underset{\sim}{\alpha} + \underset{\sim}{A}$$

Before passing to the issue of neighborhood stability, it is worth remarking that frequently we may wish to establish equilibrium for some specific value of $\underset{\sim}{S}_e$ and find what the relationships among the coefficients must be in order to achieve this. A particularly useful point to select is $\underset{\sim}{S}_e = \underset{\sim}{1}$. This makes all our species positive (a most desirable feature), and simplifies the mathematics, as we shall see. One caution; because the equations are nonlinear, dynamical solutions will not simply scale in accordance with scaling the equilibrium point, so care must be taken in "over-generalizing" the results if the conflicting species are greatly unbalanced.

Unity Equilibrium

For $\underset{\sim}{S}_e = 1$ we have

$$Q_i = \sum_{j=1}^{N} (\alpha_{ij} + a_{ij}) \quad , \quad i=1,2,\ldots N \tag{A-6}$$

and our unity equilibrium equations become

$$\dot{S}_i = - S_i\left(\sum_{j=1}^{N} \alpha_{ij} S_j\right) - \sum_{j=1}^{N} a_{ij}S_j + \sum_{j=1}^{N} (\alpha_{ij} + a_{ij}), \quad i=1,2,\ldots N \tag{A-7}$$

Note that $\dot{S}_i \Big|_{\underset{\sim}{S} = \underset{\sim}{1}} = 0$ so that unity

is indeed an equilibrium point for the system of (A-7).

Neighborhood Stability

We can study the dynamic behavior of the system very close to $\underset{\sim}{S} = \underset{\sim}{S}_e$ by a standard perturbation method. We proceed as follows:

$$\text{Let } \underset{\sim}{S} = \underset{\sim}{S}_e + \underset{\sim}{s} \tag{A-8}$$

where $\underset{\sim}{s}$ is to be a small perturbation from $\underset{\sim}{S}_e$

$$\dot{\underset{\sim}{S}} = \underset{\sim}{F}(\underset{\sim}{S}_e + \underset{\sim}{s}) + \underset{\sim}{Q} \qquad \text{so that} \tag{A-9}$$

$$\dot{\underset{\sim}{s}} = \underset{\sim}{F}(\underset{\sim}{S}_e + \underset{\sim}{s}) + \tilde{Q} \cong F(\underset{\sim}{S}_e) + \tilde{Q} + \underset{\sim}{C}\cdot\underset{\sim}{s} \tag{A-10}$$

where the elements of the matrix $\underset{\sim}{C}$ are

$$c_{ij} = \frac{\partial F_i}{\partial S_j}\Bigg|_{\underset{\sim}{S} = \underset{\sim}{S}_e} \tag{A-11}$$

Also note that $F(\underset{\sim}{S}_e) + \tilde{Q} = \underset{\sim}{0}$ (definition of an equilibrium point) so that we have the linear, first order homogeneous state equations

$$\dot{\underset{\sim}{S}} = \underset{\sim}{C}\cdot\underset{\sim}{s} \tag{A-12}$$

The solution may be found by direct integration but its properties are most easily analyzed by transformation to the LaPlace domain.*

$$p \underset{\sim}{S}(p) - \underset{\sim}{C} \underset{\sim}{S}(p) = \underset{\sim}{s}(0) \tag{A-13}$$

Here $\underset{\sim}{s}(0)$ are the initial conditions (displacement from equilibrium) and p is the Laplacian operator. We are thus interested in matrix equation

$$\underset{\sim}{S}(p) = (p \underset{\sim}{I} - \underset{\sim}{C})^{-1} \underset{\sim}{s}(0) \tag{A-14}$$

Each term of the inverse, $(p \underset{\sim}{I}-\underset{\sim}{C})^{-1}$, will contain the $\mathrm{Det}[p\underset{\sim}{I} - \underset{\sim}{C}]$ in its denominator. This determinate will be an N^{th} order polynomial, $D(p)$, if the system has N species, and thus have N roots. In the LaPlace domain then, every perturbation variable will be of the form

$$S_i(p) = \frac{N_i(p)}{D(p)} = \frac{N_i(p)}{(p-p_1)(p-p_2)..(p-p_n)} \tag{A-15}$$

where $D(p) = (p-p_1)(p-p_2)...(p-p_n)$ is the factored characteristic equation. Assuming the roots are distinct, and none correspond to poles of $N_i(p)$, the time domain solution (inverse LaPlace of (A-15)) will have the form

$$s_i(t) = \sum_{j=1}^{N} r_{ij} e^{p_j t} \quad , \quad t > 0 \tag{A-16}$$

Therefore, when the roots $\{p_j\}$ have negative real parts, lie in the left half of the complex plan (LHP), the perturbations die out with time and the system is said to have neighborhood stability. If any of the roots lie in the RHP, the system will be unstable, i.e. the smallest perturbation from equilibrium will grow without bound (in the linearized, or perturbation, system at least) and the equilibrium point is said to have neighborhood in stability. If the roots are purely complex, the

---

*Solution of (A-12) is a well established subject (See, e.g. Truxal, J.G., Introductory Systems Engineering, McGraw-Hill, 1972.

system will oscillate indefinitely and is said to be neutrally stable. (The simplest form of the prey-predator equations are neutrally stable).

The roots of the characteristic polynomial

$D(p) = \text{Det} \mid p \underset{\sim}{I} - \underset{\sim}{C} \mid$ are called the eigenvalues of the matrix $\underset{\sim}{C}$. Therefore, we learn a great deal about the stability of our system merely by studying the eigenvalues of the NXN matrix $\underset{\sim}{C}$. The matrix $\underset{\sim}{C}$, in ecology, is called the "Community Matrix". In our work, we shall refer to it as the "Conflict Matrix".

The Conflict Matrix

Recall that $\underset{\sim}{C}$ has elements

$$c_{ij} = \frac{\partial F_i}{\partial S_j} \Bigg|_{\underset{\sim}{S} = \underset{\sim}{S}_e}$$

that is, the coefficients of the first term of the Taylor series expansion of $F_i(\underset{\sim}{S})$ about $\underset{\sim}{S}_e$. Now the off-diagonal terms are easily seen from (A-3) to be

$$c_{ij} = - S_i \alpha_{ij} - a_{ij} S_{ie} \Bigg|_{\underset{\sim}{S} = \underset{\sim}{S}_e} = - S_{ie} (\alpha_{ij} + a_{ij}) \qquad \text{(A-17)}$$

The diagonal terms are

$$c_{ii} = - \sum_{j=1}^{N} \alpha_{ij} S_j - \alpha_{ii} S_i - a_{ii} S_{ie} \Bigg|_{\underset{\sim}{S} = \underset{\sim}{S}_e}$$

$$c_{ii} = - \sum_{j=1}^{N} \alpha_{ij} S_{je} - \alpha_{ii} S_{ie} - a_{ii} S_{ie}$$

$$c_{ii} = - S_{ie}(\alpha_{ii} + a_{ii}) - \sum_{j=1}^{N} \alpha_{ij} S_{je} \qquad \text{(A-18)}$$

Thus, the conflict matrix becomes, for $\underset{\sim}{S}_e = \underset{\sim}{1}$

$$\underset{\sim}{C} = -\begin{bmatrix} \alpha_{11} & \alpha_{12} & & \alpha_{1N} \\ \alpha_{21} & \alpha_{22} & \cdots\cdots & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \alpha & & & \alpha_{NN} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{N1} & \cdots\cdots & & a_{NN} \end{bmatrix} - \begin{bmatrix} \sum \alpha_{ij} & & & & \\ & \sum \alpha_{ij} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & \sum \alpha_{Nj} \end{bmatrix}$$

or simply, at the unity equilibrium point,

$$C = -\left[\underset{\sim}{\Sigma} + \alpha + \underset{\sim}{A}\right]$$

where $\underset{\sim}{\Sigma}$ is a diagonal matrix whose elements are equal to the sums of the row element of $\underset{\sim}{\alpha}$, the quadratic predation matrix.

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